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CONSERVATION OF INTEGRALS OF MOTION FOR SMALL CHANGES OF HAMILTON'S FUNCTION IN SOME CASES OF INTEGRABILITY OF THE EQUATIONS OF MOTION OF A GYROSTAT

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Motion of a gyrostat is considered. The equations of motion are written in the Hamilton form and the change in the integrals of motion in the cases of Zhukovskii and Lagrange resulting from the Hamilton function undergoing small variations is studied.

Let the mechanical system under investigation depend on a set of parameters and let it be integrable for some definite values of these parameters. Study of the motion of this system in the case when the values of the parameters are changed the system is no longer integrable, appears to be of interest. The solution of this problem involves overcoming certain fundamental difficulties connected with the problem of small denominators. In the case when the system is Hamiltonian and the changes in the values of parameters are small, these difficulties have been overcome using the method proposed by Kolmogorov and Arnol'd in [1 and 2].

Arnol'd's solution [3] of the problem of a rapidly rotating, heavy, asymmetric rigid body with a fixed point, serves to illustrate the application of this method to the rigid body dynamics.

1. Let us consider a gyrostat as a system consisting of $n+1$ rigid bodies $S_0, S_1, \ldots, S_n$, the body S_0, with a fixed point acting as a carrier to all the remaining bodies S_i which are are attached frictionlessly to S_0 at two points of the axis l_i. The axis of rotation l_i ($i = 1, 2, \ldots, n$) is the principal central axis of the body S_i, while a plane perpendicular to l_i and passing through the center of gravity C_i of S_i, is the plane of equal moments of inertia B_i. Let us attach to the body S_0 a moving coordinate system $Ox_1x_2x_3$ with its origin situated at the fixed point. Let also $\mathbf{e}_i$ (e_{i1}, e_{i2}, e_{i3}) be a unit vector directed along the axis of rotation S_i; $\mathbf{r}_i$ (r_{i1}, r_{i2}, r_{i3}) a vector directed to the center of mass of S_i from the fixed point; ω $(\omega_1, \omega_2, \omega_3)$ the angular velocity of the carrier S_0; φ_i the angle of rotation of S_i about the axis l_i: m_i the mass of S_i; A_i the moment of inertia of S_i relative to the axis l_i; and A_{ij}° the components of the inertia tensor of the carrier S_0.

We construct a Hamilton's function of the system under consideration, assuming that the bodies S_i undergo free inertial rotation. The kinetic energy of the gyrostat is

$$T = \frac{1}{2}(A_{11}\omega_1^2 + A_{22}\omega_2^2 + A_{33}\omega_3^2) + A_{23}\omega_2\omega_3 + A_{31}\omega_3\omega_1 + A_{12}\omega_1\omega_2 + \frac{1}{2}\sum_{i=1}^{n} A_i\dot{\varphi}_i^2 + \sum_{i=1}^{n} A_i(e_{i1}\omega_1 + e_{i2}\omega_2 + e_{i3}\omega_3)\dot{\varphi}_i \qquad (1.1)$$

where

$$A_{11} = A_{11}^\circ + \sum_{i=1}^{n}[m_i(r_{i2}^2 + r_{i3}^2) + B_i(e_{i2}^2 + e_{i3}^2) + A_ie_{i1}^2] \qquad (1.2)$$

$$A_{12} = A_{12}^\circ - \sum_{i=1}^{n}\left[m_ir_{i1}r_{i2} + B_ie_{i1}e_{i2} - A_ie_{i1}e_{i2}\right] \quad (1\ 2\ 3)$$

We define the spatial position of the carrier body S_0 in terms of the Euler angles ψ, ϑ, and φ, since in this case we can define completely the position of the mechanical system considered here, at any instant of time, in terms of the generalized coordinates $\psi, \vartheta, \varphi, \varphi_1, \ldots, \varphi_n$. The quantities $\omega_1, \omega_2, \omega_3$ are related to ψ, ϑ, φ in the following manner

$$\omega_1 = \dot{\psi}\sin\varphi\sin\vartheta + \dot{\vartheta}\cos\varphi \qquad (1.3)$$
$$\omega_2 = \dot{\psi}\cos\varphi\sin\vartheta - \dot{\vartheta}\sin\varphi, \quad \omega_3 = \dot{\psi}\cos\vartheta + \dot{\varphi}$$

Inserting (1.3) into (1.1) we obtain an expression for the kinetic energy in terms of the generalized coordinated and the velocity. This in turn yields the generalized impulses

$$p_\psi = \frac{\partial T}{\partial\dot{\psi}} = (A_{11}\sin\varphi\sin\vartheta + A_{12}\cos\varphi\sin\vartheta + A_{13}\cos\vartheta)\,\omega_1 + (A_{21}\sin\varphi\sin\vartheta + A_{23}\cos\varphi\sin\vartheta + A_{23}\cos\vartheta)\,\omega_2 + (A_{31}\sin\varphi\sin\vartheta + A_{32}\cos\varphi\sin\vartheta + A_{33}\cos\vartheta)\,\omega_3 + \sum_{i=1}^{n} A_i(e_{i1}\sin\varphi\sin\vartheta + e_{i2}\cos\varphi\sin\vartheta + e_{i3}\cos\vartheta)\,\dot{\varphi}_i \qquad (1.4)$$

$$p_\vartheta = \partial T/\partial\dot{\vartheta} = (A_{11}\cos\varphi - A_{12}\sin\varphi)\,\omega_1 + (A_{21}\cos\varphi - A_{22}\sin\varphi)\,\omega_2 + (A_{31}\cos\varphi - A_{32}\sin\varphi)\,\omega_3 + \sum_{i=1}^{n} A_i(e_{i1}\cos\varphi - e_{i2}\sin\varphi)\,\dot{\varphi}_i$$

$$p_\varphi = \partial T/\partial\dot{\varphi} = A_{13}\omega_1 + A_{23}\omega_2 + A_{33}\omega_3 + \sum_{i=1}^{n} A_ie_{i3}\dot{\varphi}_i$$

$$p_i = \partial T/\partial\dot{\varphi}_i = A_i\dot{\varphi}_i + A_i(e_{i1}\omega_1 + e_{i2}\omega_2 + e_{i3}\omega_3) \quad (i=1, 2, \ldots, n)$$

Insertion of the expression for $\varphi_i = \varphi_i\,(p_i)$ obtained from the last relations of (1.4) into (1.1) now yields

$$T = \frac{1}{2}\left[\left(A_{11} - \sum_{i=1}^{n} A_i e_{i1}^2\right)\omega_1^2 + \left(A_{22} - \sum_{i=1}^{n} A_i e_{i2}^2\right)\omega_2^2 + \left(A_{33} - \sum_{i=1}^{n} A_i e_{i3}^2\right)\omega_3^2 + \right.$$

$$+ \left(A_{12} - \sum_{i=1}^{n} A_i e_{i1} e_{i2}\right)\omega_1\omega_2 + \left(A_{13} - \sum_{i=1}^{n} A_i e_{i1} e_{i3}\right)\omega_1\omega_3 + \left(A_{23} - \sum_{i=1}^{n} A_i e_{i2} e_{i3}\right)\omega_2\omega_3 +$$

$$+ \frac{1}{2}\sum_{i=1}^{n} \frac{p_i^2}{A_i}$$

Let us reduce this quadratic form to its canonical form. Denoting

$$A_1 = A_{11}' - \sum_{i=1}^{n} A_i e_{i1}'^2, \quad A_2 = A_{22}' - \sum_{i=1}^{n} A_i e_{i2}'^2, \quad A_3 = A_{33}' - \sum_{i=1}^{n} A_i e_{i3}'^2$$

$$\lambda_1 = \sum_{i=1}^{n} p_i e_{i1}', \quad \lambda_2 = \sum_{i=1}^{n} p_i e_{i2}', \quad \lambda_3 = \sum_{i=1}^{n} p_i e_{i3}', \quad \Lambda = \frac{1}{2}\sum_{i=1}^{n} \frac{p_i^2}{A_i} \tag{1.5}$$

we obtain

$$T = {}^1/_2\,(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2) + \Lambda \tag{1.6}$$

Eliminating $\varphi_i^{\cdot}$, from (1.4) and taking into account (1.5) we obtain

$$p_\psi = (A_{11}\omega_1 + \lambda_1)\sin\varphi\sin\vartheta + (A_2\omega_2 + \lambda_2)\cos\varphi\sin\vartheta + (A_3\omega_3 + \lambda_3)\cos\vartheta \tag{1.7}$$

$$P_\vartheta = (A_1\omega_1 + \lambda_1)\cos\varphi - (A_2\omega_2 + \lambda_2)\sin\varphi, \quad p_\varphi = A_3\omega_3 + \lambda_3$$

The latter formula yields expressions for $\omega_1, \omega_2, \omega_3$ which, inserted into (1.6), give an expression for the kinetic energy in terms of the generalized impulses $p_\psi, p_\vartheta, p_\varphi$ generalized coordinates ψ, ϑ, φ and the quantities $\lambda_1, \lambda_2, \lambda_3$, the latter being functions of p_i. The potential energy of the system is

$$\Pi = \Gamma\,[(e_1\sin\varphi + e_2\cos\varphi)\sin\vartheta + e_3\cos\vartheta]$$

$$\left(\Gamma = Mg\,|\mathbf{r}_c|,\ M = \sum_{i=0}^{n} m_i,\ \mathbf{r}_c = |\mathbf{r}_c|\,\mathbf{e}\,(e_1, e_2, e_3) = \frac{1}{M}\sum_{i=0}^{n} m_i \mathbf{r}_i\right)$$

therefore the Hamilton function for the gyrostat can be written as follows:

$$H = \frac{1}{2A_1A_2\sin^2\vartheta}\{A_2\,[(p_\psi - p_\varphi\cos\vartheta)\sin\varphi + p_\vartheta\cos\varphi\sin\vartheta - \lambda_1\sin\vartheta]^2 + A_1\,[(p_\psi -$$

$$- p_\varphi\cos\vartheta)\cos\varphi - p_\vartheta\sin\varphi\sin\vartheta - \lambda_2\sin\vartheta]^2\} + \frac{(p_\varphi - \lambda_3)^2}{2A_3} + \Gamma\,[(e_1\sin\varphi + e_2\cos\varphi)\times$$

$$\times\sin\vartheta + e_3\cos\vartheta] + \Lambda \tag{1.8}$$

Since φ_i are cyclic coordinates, p_i as well as $\lambda_1, \lambda_2, \lambda_3$ remain constant throughout the motion. The vector λ $(\lambda_1, \lambda_2, \lambda_3)$ is called the gyrostatic moment. It characterizes the internal motions of the gyrostat and is equal to the sum of the vectors of the absolute angular momentum of the bodies S_i relative to the axes l_i respectively. When λ vanishes, the formula (1.8) becomes an expression for the Hamilton function of the heavy rigid body.

We note that the Hamilton function retains its form (1.8) also in the case when the bodies S_i rotate relative to S_0 with a constant angular velocity $\varphi_i^{\cdot 0}$. The coefficients appearing in (1.8) however assume a different meaning, namely A_1, A_2, A_3 are now the coefficients of the quadratic form reduced to the canonical form with respect to $\omega_1, \omega_2, \omega_3$, and defined by (1.2)

$$\lambda_1 = \sum_{i=1}^{n} \varphi_i^{\circ'} e_{i1}, \quad \lambda_2 = \sum_{i=1}^{n} \varphi_i^{\circ'} e_{i2}, \quad \lambda_3 = \sum_{i=1}^{n} \varphi_i^{\circ'} e_{i3}, \quad \Lambda = \frac{1}{2} \sum_{i=1}^{n} \frac{\varphi_i^{\circ' 2}}{A_i}$$

In this case the gyrostatic moment λ is equal to the sum of the angular momentum vectors of S_i relative to the axes l_i respectively.

2. From (1.8) we see that ψ is cyclic coordinate and the corresponding impulse is therefore constant

$$p_\psi = k \tag{2.1}$$

We use the integral (2.1) to decrease the number of degrees of freedom of the system to two. We make the substitution $p_\psi = k$ in the Hamilton function

$$H = H(\vartheta, \varphi, p_\vartheta, p_\varphi, k) \tag{2.2}$$

and regard k as a parameter.

Let us consider two cases in which the equations of motion containing the function (2.2) are integrable.

1. The Lagrange case: $A_1 = A_2$, $e_1 = e_2 = 0$, $\lambda_1 = \lambda_2 = 0$. The fourth integral

$$p_\varphi = m = \text{const} \tag{2.3}$$

2. The Zhukovskii case: $\Gamma = 0$. The fourth integral

$$p_\vartheta^2 + \frac{1}{\sin^2\vartheta}(p_\psi^2 + p_\varphi^2 - 2p_\varphi p_\psi \cos\vartheta) = M^2 = \text{const} \tag{2.4}$$

Let us denote in both these cases the Hamilton function by H_0, and call the motion of the gyrostat in which H_0 appears, unperturbed. Then the function

$$H = H_0 + \varepsilon H_1 \tag{2.5}$$

will define a perturbed motion (here εH_1 is a small perturbation). The following theorems hold for the perturbed motion.

T h e o r e m 2.1. If the motion of the gyrostat is described by the function (2.5) and H_0 denotes the Hamilton function in the Lagrange case, then for any $\varkappa > 0$ there exists $\varepsilon_0 > 0$ such that as soon as $0 < \varepsilon < \varepsilon_0$, $|\omega_3(t) - \omega_3(0)| < \varkappa$ for all $t \in (-\infty, \infty)$.

T h e o r e m 2.2. If the motion of the gyrostat is described by the function (2.5) and H_0 denotes the Hamiltonian function in the Zhukovskii case, then for any $\varkappa > 0$ there exists $\varepsilon_0 > 0$ such that as soon as $0 < \varepsilon < \varepsilon_0$, $|M(t) - M(0)| < \varkappa$ for all $t \in (-\infty, \infty)$.

3. Let us prove Theorem 2.1. In the Lagrange case the function H_0 is

$$H_0 = \frac{1}{2A_1}\left[p_\vartheta^2 + \left(\frac{p_\psi - p_\varphi \cos\vartheta}{\sin\vartheta}\right)^2\right] + \frac{(p_\varphi - \lambda_3)^2}{2A_3} + \Gamma c_3 \cos\vartheta + \Lambda$$

We use the canonical transformation to introduce the action-angle J_1, J_2, w_1, w_2 variables [4]. In the new variables the function H_0 depends only on J_1, and J_2. We have

$$J_1 = \int_0^{2\pi} m\, d\varphi = 2\pi m \tag{3.1}$$

$$J_2 = 2\int_{\vartheta_*}^{\vartheta^*} \left(2A_1\left[H_0 - \Gamma c_3 \cos\vartheta - \frac{(m - \lambda_3)^2}{2A_3} - \Lambda\right] - \frac{(k - m\cos\vartheta)^2}{\sin^2\vartheta}\right)^{1/2} d\vartheta$$

The quantities ϑ_*, and ϑ^* define the limits of variation of the angle of nutation ϑ. The cases in which $\vartheta = \text{const}$, are not considered. Let us set in (3.1) $\cos\vartheta = u$ and write

$$f(u) = 2A_1(1-u^2)\left[H_0 - \frac{(J_1-2\pi\lambda_3)^2}{8\pi^2A_3} - \Gamma e_3 u - \Lambda\right] - \left(k - \frac{J_1}{2\pi}u\right)^2$$

$$u_1 = \cos\vartheta_*,\ u_2 = \cos\vartheta^*$$

Then

$$J_2 = -2\int_{u_1}^{u_2} \frac{\sqrt{f(u)}}{1-u^2}\,du \tag{3.2}$$

The formula (3.2) defines H_0 as the function of J_1, and J_2. The equations of motion now give

$$\omega_1 = w_1^{\cdot} = \frac{1}{B}\int_{u_1}^{u_2} \frac{A_1(1-u^2)(J_1-2\pi\lambda_3) - A_3u(2\pi k - J_1u)}{4\pi^2A_3(1-u^2)\sqrt{f(u)}}\,du$$

$$\omega_2 = w_2 = -\frac{1}{2B}, \qquad B = A_1\int_{u_1}^{u_2}\frac{du}{\sqrt{f(u)}}$$

and the obvious expression for the frequency ratio ω_1/ω_2 can be shown to vary with J_1, J_2 when H_0 is fixed.

In the phase space the unperturbed motion can be regarded [2] as a motion with constant velocity of the representative point on a torus. The quantities ω_1, and ω_2 are the frequencies with which the corresponding angular coordinates vary on the torus, while J_1, and J_2 both remain constant.

Passing now to the perturbed motion, we shall show that the condition of nondegeneracy [2]

$$\det\left|\frac{\partial^2 H_0}{\partial J_i \partial J_j}\right| \not\equiv 0 \tag{3.3}$$

holds. It follows that the Kolmogorov theorem [1] on the conservation of motion can be applied to the function (2.5). Direct computation yields

$$\det\left|\frac{\partial^2 H_0}{\partial J_i\partial J_j}\right| = \frac{1}{16\pi^2A_1^2}\left[\int_{u_1}^{u_2}\frac{du}{\sqrt{f(u)}}\right]^{-4}\left[\int_{u_1}^{u_2}\frac{(1-u^2)\,du}{(\sqrt{f(u)})^3}\int_{u_1}^{u_2}\frac{u^2(2\pi k - J_1u)^2\,du}{4\pi^2(1-u^2)(\sqrt{f(u)})^3} - \right.$$

$$\left. - \left(\int_{u_1}^{u_2}\frac{u(2\pi k - J_1u)\,du}{2\pi(\sqrt{f(u)})^3}\right)^2 + \frac{1}{A_3}\int_{u_1}^{u_2}\frac{(1-u^2)\,du}{(\sqrt{f(u)})^3}\int_{u_1}^{u_2}\frac{A_1+(A_3-A_1)u^2}{(1-u^2)\sqrt{f(u)}}\,du\right] \not\equiv 0$$

When the value of the energy H_0 is fixed, the ratio ω_1/ω_2 varies with J_1, J_2, therefore the perturbed system has invariant tori at each energy level as well as in any neighborhood u of an arbitrary point of the phase space, provided that $\varepsilon(u)$ is sufficiently small. Since the system under consideration has two degrees of freedom, the invariant tori share the three-dimensional invariant energy level. When the initial values fall outside the invariant torus of the perturbed system, the representative point remains between the two neighboring tori during the whole motion.

The Kolmogorov theorem implies that the variation in $J_1(t)$ and $J_2(t)$ over an infinite period of time are arbitrarily small, if ε is sufficiently small. Recalling that $\omega_3 = J_1/2\pi$, we obtain the proof of Theorem 2.1.

To prove Theorem 2.2, we shall utilize the geometrical interpretation of the motion of a body when $\Gamma = 0$, due to Zhukovskii. This interpretation represents a generalization of the second Poinsot interpretation of the Euler solution for the case of gyrostat, i.e. the motion of a body is represented by rolling with slipping of a cone rigidly connected to the body, along a fixed surface. Just as in [3] we can introduce the frequencies of motion ω_1, ω_2, pass from the canonical variables ϑ, φ, p_ϑ and p_φ to the action-angle variables and reduce the problem to the proof of the condition of nondegeneracy (3.3). This condition is fulfilled in the present case, as it was shown in [3] for the case $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

4. Theorems 2.1 and 2.2 make it possible to judge the behavior of the integrals during the motion of the body, for $t \in (-\infty, \infty)$ when the parameters entering the Hamilton function undergo specific perturbations. The following theorems are valid.

Theorem 4.1. If the gyrostatic moment λ is small, then the Lagrange gyrostat moves in such a manner that the projection of its angular velocity on the third axis differs little from its initial value during the whole motion.

Theorem 4.2. If the distance between the center of gravity of the gyrostat and the fixed point is small, the magnitude of the angular momentum vector changes little throughout the whole motion.

Theorem 4.3. If the gyrostat is made to rotate rapidly, the magnitude of the angular momentum vector differs little from its initial value throughout the whole motion.

To prove Theorem 4.1 it is sufficient to set in (1.8) $A_1 = A_2$, $e_1 = e_2 = 0$, $\lambda_1 = \varepsilon\lambda_1^*$, $\lambda_2 = \varepsilon\lambda_2^*$, $\lambda_3 = \varepsilon\lambda_3^*$ and apply Theorem 2.1. The function H_0 is the Hamilton function in the Lagrange case and H_1 has the form

$$H_1 = \frac{1}{2A_1 \sin\vartheta}[\varepsilon \sin\vartheta(\lambda_1^{*2} + \lambda_2^{*2}) - 2(\lambda_1^* + \lambda_2^*)(p_\vartheta \cos\varphi \sin\vartheta + (p_\psi - p_\varphi \cos\vartheta)\sin\varphi] + \\ + \frac{\varepsilon\lambda_3^{*2} - 2p_\varphi\lambda_3^*}{2A_3} + \varepsilon\Lambda^*$$

Setting in (1.8) $\Gamma = \varepsilon\Gamma^*$ we find that the function H can be written in the form (2.5), H_0 is the Hamilton function in the Zhukovskii case and

$$H_1 = \Gamma^* \, [(e_1 \sin\varphi + e_2 \cos\varphi)\sin\vartheta + e_3 \cos\vartheta]$$

Applying now Theorem 2.2 we obtain the proof of Theorem 4.2. Theorem 4.3 follows from Theorem 4.2 as the problem of rapid rotation of a gyrostat is mathematically equivalent to the problem of motion of a gyrostat in a weak attraction field.

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